

Regularity for Harmonic - Einstein Equation

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Abstract

We establish a regularity theorem for the Harmonic - Einstein Equation. As a byproduct of the local regularity, we also have a compactness theorem on Harmonic - Einstein equation. The method is mainly the Moser iteration technique which has been used and developed by [BKN89], [Tian90], [TV05a] and others.

1 Introduction

In this paper, we will consider the degeneration of Harmonic - Einstein equation, which is a generalization the Einstein equation. Another motivation is that John Lott showed that the expanding soliton equation on the space with the simplest type of Nil structure can be reduced to the Harmonic - Einstein equation, while such kind of soliton appeared in the long time limit of type - III Ricci flow.

Theorem 1.1. [Lott07] Let $(\overline{M}, \overline{g})$ be the total space of a flat \mathbb{R}^N - vector bundle over a Riemannian manifold (M, g) , with flat Riemannian metrics on the fibers. Suppose that the fiberwise volume forms are preserved by the flat connection. Let V be the fiberwise radial vector field $\frac{1}{2} \sum_{i=1}^N x^i \frac{\partial}{\partial x_i}$. Then the expanding soliton equation on \overline{M}

$$\overline{Ric} + \frac{1}{2} \mathcal{L}_V \overline{g} + \frac{1}{2} \overline{g} = 0$$

becomes the equation for a harmonic map

$$G : (M, g) \rightarrow (SL(N, \mathbb{R})/SO(N), \langle \cdot, \cdot \rangle) \quad (1.1)$$

along the equation

$$Ric - \frac{1}{4} \langle dG, dG \rangle + \frac{1}{2} g = 0 \quad (1.2)$$

on M , where $\langle \cdot, \cdot \rangle_G = Tr(G^{-1} dG G^{-1} dG)$ is the usual metric on the symmetric space $SL(N, \mathbb{R})/SO(N)$.

Remark 1. There is an algebraic description of Symmetric space $SL(N, \mathbb{R})/SO(N)$: $sl(N) = \{X : tr X = 0\} \simeq \mathfrak{h} \oplus \mathfrak{so}(N)$, where \mathfrak{h} is the symmetric part, which can be identified with the tangent space of $SL(N, \mathbb{R})/SO(N)$. On \mathfrak{h} we have the usual Euclidean metric, the involution L is $-id$ on \mathfrak{h} and id on $\mathfrak{so}(N)$, namely, $L(X) = -X^t$. Consequently, the curvature is $Rm(X, Y, Z, W) = -\langle [X, Y], [Z, W] \rangle$. In particular, the sectional curvature is nonpositive, which is crucial in our result.

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Definition 1.2. Let $G : (M, g) \longrightarrow (N, h)$ be a map between two Riemannian manifolds, with the notation in [Lott07], we will call the local version of the equations

$$\begin{cases} 0 = Ric(g) - \langle dG, dG \rangle - \lambda g \\ 0 = \Delta_{g,h} G \end{cases} \quad (1.3)$$

or in local coordinates,

$$\begin{cases} 0 = Ric(g)_{\alpha\beta} - h_{ij} G_{,\alpha}^i G_{,\beta}^j - \lambda g_{\alpha\beta} \\ 0 = g^{\alpha\beta} G_{,\alpha\beta}^i - g^{\alpha\beta} G_{,\alpha}^j G_{,\beta}^k \Gamma(h)_{jk}^i \end{cases} \quad (1.4)$$

to be **Harmonic-Einstein** equation.

Since any Riemannian metric satisfies $\Delta Rm = \nabla^2 Ric + Rm * Rm$, combine with the equation (1.3), (M, g, G) satisfies a coupled Elliptic system, together with uniform Sobolev constant C_S , one can prove an ϵ - regularity theorem.

Theorem 1.3. (ϵ - regularity) Assume (M, g, G) satisfies the Harmonic-Einstein equation (1.3), and (N, h) has nonpositive sectional curvature, $\lambda = 0$ or -1 . Let $B(0, r)$ be a geodesic ball around $0 \in M$, C_S be the Sobolev constant on $B(0, r)$, and $k \in \mathbb{N}$. Then there exists a constant $\epsilon = \epsilon(C_S, n)$ such that if

$$\left\{ \int_{B(0,r)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \leq \epsilon,$$

then

$$\begin{aligned} \sup_{B(0, \frac{r}{2})} |\nabla^k G| &\leq \frac{C}{r^k} \left(\left\{ \int_{B(0,r)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \{r^{2-n} \int_{B(0,r)} |dG|^2\}^{\frac{1}{2}} \right), \\ \sup_{B(0, \frac{r}{2})} |\nabla^k Rm| &\leq \frac{C}{r^{k+2}} \left(\left\{ \int_{B(0,r)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \{r^{2-n} \int_{B(0,r)} |dG|^2\}^{\frac{1}{2}} \right), \end{aligned}$$

where $C = C(C_S, k, n)$.

At this point, the author would like to point out the method here can also be used to prove the ϵ - regularity theorem on the system $\Delta Ric = Rm * Ric$. The four dimensional case, namely, Bach flat metric with constant scalar curvature, has been established by Tian and Viaclovsky [TV05a], and the higher dimension case has been proved by Chen and Weber [CW11]. The main idea are all similar, but one will see our iteration process is different.

As a byproduct of the ϵ - regularity, we obtain a convergence theorem for Harmonic - Einstein equation, which is similar to the compactness on harmonic maps [SaUh81], Yang - Mills connections [Uhlenbeck82a], Einstein metrics [Anderson89], [BKN89], [Tian90], [Nakajima94], and more recently Bach flat metric with constant scalar curvature [TV05b], [AAJV11], Kähler Ricci soliton [CS07], extremal Kähler metric [CW11].

Theorem 1.4. (Compactness) Let (g_i, G_i) satisfy the Harmonic - Einstein equation (1.3) over a sequence of 4 - dimensional compact manifolds M_i , respectively. Assume (M_i, g_i) satisfy:

$$\text{Euler number } \mathcal{X}(M_i) \leq X, \text{ Diam}(M_i, g_i) \leq D, \text{ Vol}(M_i, g_i) \geq V,$$

and $G_i : (M_i, g_i) \longrightarrow (N, h)$ with finite energy

$$E(G_i, g_i) := \int_{M_i} |dG_i|^2 \leq E,$$

where X, E, D, V are constants which are independent of i . We also assume (N, h) has nonpositive sectional curvature and $\lambda = 0$ or -1 . Then there exists a subsequence $\{j\} \subset \{i\}$ satisfies the following properties:

1. $\{M_j, g_j, G_j\}$ converges to a complete metric space M_∞ in the following sense: If we remove a finite set $\mathcal{S} = \{b_1, \dots, b_m\} \subset M_\infty$ with $m \leq m(n, X, D, V)$, a C^∞ manifold structure is defined and also a smooth pair (g_∞, G_∞) satisfies the Harmonic - Einstein equation over the punctured set $M_\infty \setminus \mathcal{S}$. Moreover, there exists a (into) diffeomorphism $F_j : M_\infty \setminus \mathcal{S} \hookrightarrow M_i$ such that $(F_j^* g_j, F_j^* G_j)$ converges to (g_∞, G_∞) in the $C^\infty(M_\infty \setminus \mathcal{S})$ topology.
2. The manifold structure and the pair (g_∞, G_∞) on $M_\infty \setminus \mathcal{S}$ extend to the whole of M which satisfies the Harmonic - Einstein equation (1.3) over a Riemannian orbifold.

Definition 1.5. By (g, G) satisfies the Harmonic - Einstein equation (1.3) over a Riemannian orbifold, (M, g) , we mean:

1. There exists a finite set $\mathcal{S} = \{b_1, \dots, b_m\} \subset M$, such that $M \setminus \mathcal{S}$ is a C^∞ manifold and the restriction of (g, G) satisfies the smooth Harmonic - Einstein equation (1.3).
2. For each singular point $b_k \in \mathcal{S}$, there exists a neighborhood $U_k \subset M$ such that $U_k \setminus \{x_k\}$ is diffeomorphic to $B^n \setminus \{0\}/\Gamma_k$, where $B^n \in \mathbb{R}^n$ is a n - dimensional unit ball and $\Gamma_k \subset O(n)$ is a finite subgroup acting freely on $B^n \setminus \{0\}/\Gamma$. If we lift (g, G) to $B^n \setminus \{0\}$, it extends smoothly across the singular point 0 and satisfies the Harmonic - Einstein equation (1.3) over B^n .

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2 Bochner Identities

Let $f : (M, g) \longrightarrow (N, h)$ be a map between two Riemannian manifolds, the differential of f is

$$df = \frac{\partial f^i}{\partial x^\alpha} dx^\alpha \otimes \frac{\partial}{\partial f^i} \in \Gamma(T^*M \otimes f^{-1}TN).$$

Now we would like to establish the Bochner - type identity on the extended bundle $\Omega^p T^*M \otimes \Omega^q f^{-1}TN$ over M with respect to the induced metric $g \otimes f^*h$ and induced connection $\nabla^M \otimes f^*\nabla^N$. These kind identities should be well known by experts.

Lemma 2.1. Let $f : (M, g) \longrightarrow (N, h)$ be a map between two Riemannian manifolds, then we have the well known Bochner formula, for instance, see [EL78] and [SY94].

$$\Delta \frac{1}{2} |df|^2 = |\nabla df|^2 + \langle \nabla \Delta f, \Delta f \rangle - \sum_{\alpha, \beta} Rm^N(f_* e_\alpha, f_* e_\beta, f_* e_\alpha, f_* e_\beta) + \sum_i Ric^M(f^* \theta_i, f^* \theta_i)$$

where $\{e_\alpha\}$ is an orthonormal basis for TM , $\{\theta_i\}$ is an orthonormal basis for T^*N . More generally, we also have the Bochner type identities for commutation of covariant derivatives up to order k :

$$\begin{aligned} \Delta \nabla^k f &= \nabla^k \Delta f + \sum_{i=0}^{k-1} \nabla^i Rm^M * \nabla^{k-i} f \\ &\quad + \sum_{p=3}^{k+2} \sum_{i_1 + \dots + i_p = k-p+2} \nabla^{p-3} Rm^N * \nabla^{i_1+1} f * \dots * \nabla^{i_p+1} f \end{aligned} \quad (2.1)$$

In particular, if the target manifold is symmetric, i.e. $\nabla Rm^N = 0$, then we can drop the terms which involve the derivative of Rm^N in the above expression:

$$\Delta \nabla^k f = \nabla^k \Delta f + \sum_{i=0}^{k-1} \nabla^i Rm^M * \nabla^{k-i} f + \sum_{i=0}^{k-1} \sum_{p+q=i} Rm^N(\nabla^{p+1} f, \nabla^{q+1} f) \nabla^{k-i} f$$

Proof. In order to simplify the computation, we choose normal coordinates at x and $f(x)$ respectively, namely,

$$g_{\alpha\beta}(x) = \delta_{\alpha\beta}, g_{\alpha\beta,\gamma}(x) = 0; \quad h_{ij}(f(x)) = \delta_{ij}, h_{ij,k}(f(x)) = 0.$$

Therefore, we only have to take the second and up derivatives of the metric into account, and these will turn into the curvature terms. First, let us compute the commutation of covariant derivatives up to three order directly:

$$\begin{aligned} & \nabla^3 f\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\gamma}\right) = \nabla_{\frac{\partial}{\partial x^\alpha}}(\nabla^2 f\left(\frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\gamma}\right)) \\ &= \nabla_{\frac{\partial}{\partial x^\alpha}}\left(\frac{\partial^2 f^i}{\partial x^\beta \partial x^\gamma} \frac{\partial}{\partial f^i} + \frac{\partial f^i}{\partial x^\gamma} \frac{\partial f^j}{\partial x^\beta} \nabla_{\frac{\partial}{\partial f^j}} \frac{\partial}{\partial f^i} - df\left(\nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma}\right)\right) \\ &= \frac{\partial^3 f^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \frac{\partial}{\partial f^i} + \frac{\partial f^i}{\partial x^\gamma} \frac{\partial f^j}{\partial x^\beta} \frac{\partial f^k}{\partial x^\alpha} \nabla_{\frac{\partial}{\partial f^k}} \nabla_{\frac{\partial}{\partial f^j}} \frac{\partial}{\partial f^i} - df\left(\nabla_{\frac{\partial}{\partial x^\alpha}} \nabla_{\frac{\partial}{\partial x^\beta}} \frac{\partial}{\partial x^\gamma}\right) \\ &= \frac{\partial^3 f^i}{\partial x^\alpha \partial x^\beta \partial x^\gamma} \frac{\partial}{\partial f^i} + \frac{\partial f^i}{\partial x^\gamma} \frac{\partial f^j}{\partial x^\beta} \frac{\partial f^k}{\partial x^\alpha} \left(\nabla_{\frac{\partial}{\partial f^i}} \nabla_{\frac{\partial}{\partial f^k}} \frac{\partial}{\partial f^j} - Rm^N\left(\frac{\partial}{\partial f^k}, \frac{\partial}{\partial f^i}\right) \frac{\partial}{\partial f^j}\right) \\ &\quad - df\left(\nabla_{\frac{\partial}{\partial x^\gamma}} \nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} - Rm^M\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\gamma}\right) \frac{\partial}{\partial x^\beta}\right) \\ &= \nabla^3 f\left(\frac{\partial}{\partial x^\gamma}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) - Rm^N\left(\frac{\partial f^k}{\partial x^\alpha} \frac{\partial}{\partial f^k}, \frac{\partial f^i}{\partial x^\gamma} \frac{\partial}{\partial f^i}\right) \frac{\partial f^j}{\partial x^\beta} \frac{\partial}{\partial f^j} + df\left(Rm^M\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\gamma}\right) \frac{\partial}{\partial x^\beta}\right) \end{aligned}$$

taking trace with respect to α and β , we obtain

$$\Delta \nabla f = \nabla \Delta f + Rm^N(f_*(\frac{\partial}{\partial x^\alpha}), df)f_*(\frac{\partial}{\partial x^\alpha}) + df(Ric^M)$$

Consequently, we get the Bochner formula:

$$\begin{aligned} \Delta \frac{1}{2} |df|^2 &= |\nabla df|^2 + \langle \Delta \nabla f, \nabla f \rangle \\ &= |\nabla df|^2 + \langle \nabla \Delta f, \nabla f \rangle - Rm^N(f_*(\frac{\partial}{\partial x^\alpha}), f_*(\frac{\partial}{\partial x^\gamma}), f_*(\frac{\partial}{\partial x^\alpha}), f_*(\frac{\partial}{\partial x^\gamma})) \\ &\quad + \langle df(Ric^M(\frac{\partial}{\partial x^\gamma})), df(\frac{\partial}{\partial x^\gamma}) \rangle \end{aligned}$$

Therefore, we proved the case $k = 1$. Now let us assume the expression (2.1) holds for $k - 1$, and we will prove the case k .

More generally, on the extended bundle $\Omega^p T^* M \otimes \Omega^q f^{-1} T N$ over M with respect to the induced connection $\nabla^M \otimes f^* \nabla^N$, for

$$T = T_{\alpha_1, \dots, \alpha_p}^{i_1, \dots, i_q} dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_p} \otimes \frac{\partial}{\partial f^1} \otimes \dots \otimes \frac{\partial}{\partial f^q} \in \Omega^p T^* M \otimes \Omega^q f^{-1} T N,$$

with abuse of notation:

$$\nabla_{\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}}^2 T - \nabla_{\frac{\partial}{\partial x^\beta}, \frac{\partial}{\partial x^\alpha}}^2 T = R\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) T$$

where

$$\begin{aligned} Rm(T) &= \sum_{i=1}^p T_{\alpha_1, \dots, \alpha_p}^{i_1, \dots, i_q} dx^{\alpha_1} \otimes \dots \otimes Rm^M(dx^{\alpha_i}) \otimes \dots \otimes dx^{\alpha_p} \otimes \frac{\partial}{\partial f^1} \otimes \dots \otimes \frac{\partial}{\partial f^q} \\ &\quad + \sum_{j=1}^q T_{\alpha_1, \dots, \alpha_p}^{i_1, \dots, i_q} dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_p} \otimes \frac{\partial}{\partial f^1} \otimes \dots \otimes Rm^N(df, df)\left(\frac{\partial}{\partial f^j}\right) \otimes \dots \otimes \frac{\partial}{\partial f^q} \\ &= (Rm^M + Rm^N(df, df)) * T \end{aligned}$$

then

$$\begin{aligned}
\nabla Rm(T) &= \nabla \left((Rm^M + Rm^N(df, df)) * T \right) \\
&= (\nabla Rm^M + Rm^N(\nabla df, df) + df \otimes \nabla Rm^N(df, df)) * T \\
&\quad + (Rm^M + Rm^N(df, df)) * \nabla T
\end{aligned} \tag{2.2}$$

With this notation, let us compute the $\Delta \nabla^k f$,

$$\begin{aligned}
&\Delta \nabla^k f \left(\frac{\partial}{\partial x^{\beta_1}}, \dots, \frac{\partial}{\partial x^{\beta_k}} \right) \\
&= \nabla^{k+2} f \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \dots, \frac{\partial}{\partial x^{\beta_k}} \right) \\
&= \nabla_{\frac{\partial}{\partial x^\alpha}} \left(\nabla^{k+1} f \left(\frac{\partial}{\partial x^{\beta_1}}, \frac{\partial}{\partial x^\alpha}, \dots \right) + Rm(\nabla^{k-1} f) \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \dots \right) \right) \\
&= \nabla^{k+2} f \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \frac{\partial}{\partial x^\alpha}, \dots \right) + \nabla(Rm(\nabla^{k-1} f)) \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \dots \right) \\
&= \nabla^{k+2} f \left(\frac{\partial}{\partial x^{\beta_1}}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\alpha}, \dots \right) + Rm(\nabla^k f) \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \frac{\partial}{\partial x^\alpha}, \dots \right) \\
&\quad + \nabla(Rm(\nabla^{k-1} f)) \left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^{\beta_1}}, \dots \right)
\end{aligned}$$

From the above calculation, by induction and formula (2.2),

$$\begin{aligned}
\Delta \nabla^k f &= \nabla \Delta \nabla^{k-1} f + Rm(\nabla^k f) + \nabla(Rm(\nabla^{k-1} f)) \\
&= \nabla \nabla^{k-1} \Delta f + \sum_{i=0}^{k-2} \nabla \left(\nabla^i Rm^M * \nabla^{k-i-1} f \right) + (Rm^M + Rm^N(df, df)) * \nabla^k f \\
&\quad + \sum_{p=3}^{k+1} \sum_{i_1+\dots+i_p=k-p+1} \nabla \left(\nabla^{p-3} Rm^N * \nabla^{i_1+1} f * \dots * \nabla^{i_p+1} f \right) \\
&\quad + (\nabla Rm^M + Rm^N(\nabla df, df) + df \otimes \nabla Rm^N(df, df)) * \nabla^{k-1} f \\
&= \nabla^k \Delta f + \sum_{i=0}^{k-1} \nabla^i Rm^M * \nabla^{k-i} f \\
&\quad + \sum_{p=3}^{k+2} \sum_{i_1+\dots+i_p=k-p+2} \nabla^{p-3} Rm^N * \nabla^{i_1+1} f * \dots * \nabla^{i_p+1} f
\end{aligned}$$

Thus the formula (2.1) holds for k .

In particular, if the target manifold is a symmetric space, i.e. $\nabla Rm^N = 0$, therefore we can drop the terms which involve the derivative of Rm^N in the above expression. With this simplification, we have the Bochner identity:

$$\Delta \nabla^k f = \nabla^k \Delta f + \sum_{i=0}^{k-1} \nabla^i Rm^M * \nabla^{k-i} f + \sum_{i=0}^{k-1} \sum_{i_1+i_2=i} Rm^N * \nabla^{i_1+1} f * \nabla^{i_2+1} f * \nabla^{k-i} f$$

One can see the expression (2.1) is homogeneous with respect to covariant derivation. Without the symmetric condition, i.e. $\nabla Rm = 0$, there will be more terms occur, which is higher order in f but same order with respect to derivation. \square

Lemma 2.2. Suppose $G : (M, g) \longrightarrow (N, h)$ satisfies the Harmonic - Einstein metric (1.3), then we have the coupled system for the full curvature tensor and harmonic map:

$$\Delta \nabla^k Rm = Rm * \nabla^k Rm + \sum_{i=0}^{k+2} \nabla^{i+1} G * \nabla^{k+3-i} G + \sum_{i=1}^{k-1} \nabla^i Rm * \nabla^{k-i} Rm \quad (2.3)$$

$$\Delta \nabla^k G = \sum_{i=0}^{k-1} \nabla^i Rm * \nabla^{k-i} G + \sum_{p=3}^{k+2} \sum_{i_1+\dots+i_p=k-p+2} \nabla^{i_1+1} G * \dots * \nabla^{i_p+1} G \quad (2.4)$$

Proof. First, let us recall that the full Riemannian curvature of any Riemannian metric satisfies the following equation

$$\Delta Rm = \nabla^2 Ric + Rm * Rm$$

and the well known Bochner formula on M , see [Besse87]:

$$\begin{aligned} \Delta \nabla^k Rm &= \nabla^k \Delta Rm + \sum_{i=0}^k \nabla^i Rm * \nabla^{k-i} Rm \\ &= \nabla^{k+2} Ric + \sum_{i=0}^k \nabla^i Rm * \nabla^{k-i} Rm \end{aligned}$$

Consequently, coupled with the Harmonic - Einstein equation (1.3), then Ricci curvature is related to dG ,

$$Ric = \lambda g + \langle dG, dG \rangle$$

Replacing Ric term in the above expression, we have (2.3),

$$\begin{aligned} \Delta \nabla^k Rm &= \nabla^{k+2} (dG * dG) + \sum_{i=0}^k \nabla^i Rm * \nabla^{k-i} Rm \\ &= \sum_{i=0}^{k+2} \nabla^{i+1} G * \nabla^{k+3-i} G + \sum_{i=0}^k \nabla^i Rm * \nabla^{k-i} Rm \end{aligned}$$

Since G is a harmonic map, (2.4) follows from lemma 2.1. Moreover, we can drop the covariant derivatives of Rm^N , since the target metric does not deform any more. \square

3 Local Regularity

Now let us establish the ϵ - Regularity for Harmonic - Einstein equation (1.3) by divided the proof into several lemmas.

Lemma 3.1. Let $G : (M, g) \longrightarrow (N, h)$ satisfies the Harmonic - Einstein equation (1.3). Assume (N, h) has nonpositive sectional curvature and (M, g) has bounded Sobolev constant C_S . Then we have

$$\sup_{B_{\frac{R}{2}}(0)} |\nabla G|^2 \leq \frac{C}{R^n} \int_{B_R(0)} |\nabla G|^2. \quad (3.1)$$

In other words, $|\nabla G|$ is bounded. Therefore, the *Ricci* curvature is two sided bounded, and consequently, the volume of geodesic ball is comparable with Euclidean Ball.

Proof. Since (N, h) has nonpositive sectional curvature, and $\langle dG, dG \rangle$ is nonnegative, then by the Bochner formula in lemma 2.1, we obtain:

$$\begin{aligned} \frac{1}{2} \Delta |dG|^2 &= |\nabla \nabla f|^2 - \sum_{\alpha, \beta} Rm^N(G_* e_\alpha, G_* e_\beta, G_* e_\alpha, G_* e_\beta) + \sum_i Ric^M(G^* \theta_i, G^* \theta_i) \\ &\geq \sum_i Ric^M(G^* \theta_i, G^* \theta_i) \\ &\geq \lambda |dG|^2 \end{aligned}$$

With above equation in hand, by elliptic Moser iteration with uniform Sobolev constant, the L^∞ norm of $|dG|^2$ can be bounded by L^2 of $|dG|^2$. Actually, this technique will be used through our paper for more general tensors, so will not give the detail here. With more effort, one can also iterate by virtue of a well known way, then L^∞ norm can be bounded by L^p norm of $|dG|^2$ for any $p > 0$. Now, we have L^1 - norm of $|dG|^2$, so we have,

$$\sup_{B_{\frac{1}{2}}(0)} |dG|^2 \leq C \int_{B_1(0)} |dG|^2.$$

For more details, see [GT83], [SY94], [Simon96]. \square

From now on, we will denote $\gamma = \frac{n}{n-2}$ through out this paper. ϕ will be a cut off function with $supp \phi \subset B(0, r)$, and $\phi \equiv 1$ on $B(0, \tau)$ with $|\nabla \phi| \leq \frac{2}{r-\tau}$. The estimation below will be affected by different choices of r and τ , therefore we will choose proper cut off function with respect to our purpose.

Lemma 3.2. (Iteration I: $\|T\|_{L^{p\gamma}}$ estimation from ΔT with $\|T\|_{L^p}$) Let T be a tensor, then

$$\left\{ \int (\phi |T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \leq C \left(\int |\nabla \phi|^2 |T|^p + p \phi^2 |T|^{p-2} \langle T, -\Delta T \rangle \right) \quad (3.2)$$

Proof. First, let us do some basic calculation. With Kato inequality $|\nabla |T|| \leq |\nabla T|$, then

$$\begin{aligned} -|T| \Delta |T| &= -\frac{1}{2} \Delta |T|^2 + |\nabla |T||^2 \\ &= -\langle T, \Delta T \rangle - (|\nabla T|^2 - |\nabla |T||^2) \\ &\leq -\langle T, \Delta T \rangle \end{aligned}$$

Consequently, when $p \geq 2$,

$$\begin{aligned} -\Delta |T|^{\frac{p}{2}} &= -\frac{p}{2} |T|^{\frac{p}{2}-1} \Delta |T| - \frac{p}{2} \left(\frac{p}{2} - 1 \right) |T|^{\frac{p}{2}-2} |\nabla |T||^2 \\ &\leq -\frac{p}{2} |T|^{\frac{p}{2}-2} \langle T, \Delta T \rangle \end{aligned}$$

Moreover, from the Schwarz inequality we obtain

$$\begin{aligned} \phi^2 |\nabla |T|^{\frac{p}{2}}|^2 &= \text{div}(\phi^2 |T|^{\frac{p}{2}} \nabla |T|^{\frac{p}{2}}) - \phi^2 |T|^{\frac{p}{2}} \Delta |T|^{\frac{p}{2}} - 2 \langle \phi \nabla |T|^{\frac{p}{2}}, \nabla \phi |T|^{\frac{p}{2}} \rangle \\ &\leq \text{div}(\phi^2 |T|^{\frac{p}{2}} \nabla |T|^{\frac{p}{2}}) - \frac{p}{2} \phi^2 |T|^{p-2} \langle T, \Delta T \rangle + \delta \phi^2 |\nabla |T|^{\frac{p}{2}}|^2 + \frac{1}{\delta} |\nabla \phi|^2 |T|^p \end{aligned}$$

Taking $\delta = \frac{1}{2}$, then the term $\delta \phi^2 |\nabla |T|^{\frac{p}{2}}|^2$ can be absorbed by the left,

$$\phi^2 |\nabla |T|^{\frac{p}{2}}|^2 \leq 2 \text{div}(\phi^2 |T|^{\frac{p}{2}} \nabla |T|^{\frac{p}{2}}) - p \phi^2 |T|^{p-2} \langle T, \Delta T \rangle + 4 |\nabla \phi|^2 |T|^p$$

By Sobolev inequality,

$$\begin{aligned}
\left\{ \int (\phi|T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C_S \int |\nabla(\phi|T|^{\frac{p}{2}})|^2 \\
&\leq C \left(\int |\nabla\phi|^2 |T|^p + \int \phi^2 |\nabla|T|^{\frac{p}{2}}|^2 \right) \\
&\leq C \left(\int |\nabla\phi|^2 |T|^p + p\phi^2 |T|^{p-2} \langle T, -\Delta T \rangle \right)
\end{aligned}$$

Actually, this is nothing but the Moser iteration relation which is generalized to tensor. \square

Lemma 3.3. ($\|\nabla T\|_{L^2}$ estimation from ΔT) Let T be a tensor, then

$$\int \phi^2 |\nabla T|^2 \leq C \left(\int \phi^2 \langle T, -\Delta T \rangle + \int |\nabla\phi|^2 |T|^2 \right) \quad (3.3)$$

In particular, if

$$\Delta T = Rm * T + cT + \nabla X + Y$$

where c is some constant, X, Y are tensors. Then there exists a constant $\epsilon = \epsilon(C_S, n)$, if

$$\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{n}{2}} \leq \epsilon,$$

then we have

$$\int \phi^2 |\nabla T|^2 \leq C \left(\int \phi^2 |X|^2 + \int \phi^2 |Y|^2 + \int (\phi^2 + |\nabla\phi|^2) |T|^2 \right) \quad (3.4)$$

Proof. As did in lemma 3.2, we have,

$$\begin{aligned}
\phi^2 |\nabla T|^2 &= \operatorname{div}(\phi^2 \langle T, \nabla T \rangle) - \phi^2 \langle T, \Delta T \rangle - 2\langle \phi \nabla T, \nabla \phi T \rangle \\
&\leq \operatorname{div}(\phi^2 \langle T, \nabla T \rangle) + \phi^2 \langle T, -\Delta T \rangle + \delta \phi^2 |\nabla T|^2 + \frac{1}{\delta} |\nabla\phi|^2 |T|^2
\end{aligned}$$

Therefore

$$\int \phi^2 |\nabla T|^2 \leq C \left(\int \phi^2 \langle T, -\Delta T \rangle + \int |\nabla\phi|^2 |T|^2 \right)$$

In particular, if

$$\Delta T = Rm * T + cT + \nabla X + Y$$

then the Laplacian term can be reduced to:

$$\begin{aligned}
\phi^2 \langle T, -\Delta T \rangle &= \phi^2 \langle T, -Rm * T - cT - \nabla X - Y \rangle \\
&= -\operatorname{div}(\phi^2 \langle T, X \rangle) + \phi^2 \langle \nabla T, X \rangle + 2\phi \langle \nabla\phi \otimes T, X \rangle \\
&\quad + \phi^2 \langle T, -Rm * T \rangle - \phi^2 \langle T, Y \rangle - c\phi^2 |T|^2 \\
&\leq -\operatorname{div}(\phi^2 \langle T, X \rangle) + \delta \phi^2 |\nabla T|^2 + C(\phi^2 |X|^2 + |\nabla\phi|^2 |T|^2 \\
&\quad + \phi^2 |Rm| |T|^2 + \phi^2 |T|^2 + \phi^2 |Y|^2)
\end{aligned}$$

and hence

$$\begin{aligned}
\phi^2 |\nabla T|^2 &\leq \operatorname{div}(\phi^2 \langle T, \nabla T \rangle) - \operatorname{div}(\phi^2 \langle T, X \rangle) \\
&\quad + C(\phi^2 |Rm| |T|^2 + \phi^2 |X|^2 + \phi^2 |Y|^2 + (\phi^2 + |\nabla\phi|^2) |T|^2)
\end{aligned}$$

By Sobolev inequality

$$\begin{aligned}
\left\{ \int (\phi|T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int |\nabla\phi|^2 |T|^2 + \int \phi^2 |\nabla T|^2 \right) \\
&\leq C \left(\int \phi^2 |Rm| |T|^2 + \phi^2 |X|^2 + \phi^2 |Y|^2 + (\phi^2 + |\nabla\phi|^2) |T|^2 \right)
\end{aligned}$$

Moreover, if

$$C\left\{\int_{B(0,1)} |Rm|^{\frac{n}{2}}\right\}^{\frac{n}{2}} \leq \frac{1}{2},$$

then the first term

$$C \int \phi^2 |Rm| |T|^2 \leq C \left\{\int |Rm|^{\frac{n}{2}}\right\}^{\frac{2}{n}} \left\{\int (\phi|T|)^{2\gamma}\right\}^{\frac{1}{\gamma}} \leq \frac{1}{2} \left\{\int (\phi|T|)^{2\gamma}\right\}^{\frac{1}{\gamma}}$$

can be absorbed by the left, thus we obtain

$$\left\{\int (\phi|T|)^{2\gamma}\right\}^{\frac{1}{\gamma}} \leq C \left(\int \phi^2 |X|^2 + \int \phi^2 |Y|^2 + \int (\phi^2 + |\nabla \phi|^2) |T|^2\right)$$

which in turn implies

$$\int \phi^2 |\nabla T|^2 \leq C \left(\int \phi^2 |X|^2 + \int \phi^2 |Y|^2 + \int (\phi^2 + |\nabla \phi|^2) |T|^2\right)$$

□

Lemma 3.4. (Iteration II: $\|\nabla T\|_{L^{p\gamma}}$ estimation from $\Delta \nabla T$ with $\|\nabla T\|_{L^p}$) For any $p \geq 2$, there exist a constant $\epsilon = \epsilon(C_S, n)$, if

$$\left\{\int_{B(0,1)} |Rm|^{\frac{n}{2}}\right\}^{\frac{n}{2}} \leq \frac{\epsilon}{p},$$

then

$$\begin{aligned} \left\{\int (\phi|\nabla T|^{\frac{p}{2}})^{2\gamma}\right\}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla T|^{p-2} |\Delta T|^2 + \int \phi^2 |\nabla T|^{p-2} |Rm|^2 |T|^2\right. \\ &\quad \left.+ \int |\nabla \phi|^2 |\nabla T|^p\right) \end{aligned} \quad (3.5)$$

In our case, $Rm = Rm^M + Rm^N(dG, dG)$, since $|\nabla G|^2 |Rm^N|$ is bounded, as explained in the proof, the above inequality holds if we refer Rm as Rm^M .

Proof. Similarly, replacing T by ∇T in lemma 3.2, we have

$$\phi^2 |\nabla |\nabla T|^{\frac{p}{2}}|^2 \leq \operatorname{div}(\phi^2 |\nabla T|^{\frac{p}{2}} \nabla |\nabla T|^{\frac{p}{2}}) - p \phi^2 |\nabla T|^{p-2} \langle \nabla T, \Delta \nabla T \rangle + 4 |\nabla \phi|^2 |\nabla T|^p$$

By Bochner formula

$$\Delta \nabla T = \nabla \Delta T + \nabla (Rm * T) + Rm * \nabla T$$

and the we have

$$\begin{aligned} -p \phi^2 |\nabla T|^{p-2} \langle \nabla T, \Delta \nabla T \rangle &= -p \phi^2 |\nabla T|^{p-2} \langle \nabla T, \nabla \Delta T + \nabla (Rm * T) + Rm * \nabla T \rangle \\ &= p \{-\operatorname{div}(\phi^2 |\nabla T|^{p-2} \langle \nabla T, \Delta T + Rm * T \rangle) \\ &\quad + \phi^2 |\nabla T|^{p-2} \langle \Delta T, \Delta T + Rm * T \rangle \\ &\quad + (p-2) \phi^2 |\nabla T|^{p-3} \langle \nabla |\nabla T| \otimes \nabla T, \Delta T + Rm * T \rangle \\ &\quad + 2 \phi |\nabla T|^{p-2} \langle \nabla \phi \otimes \nabla T, \Delta T + Rm * T \rangle \\ &\quad + \phi^2 |\nabla T|^{p-2} \langle \nabla T, Rm * \nabla T \rangle\} \end{aligned}$$

For the term which involves second covariant derivative of T , apply the Schwarz inequality,

$$\begin{aligned} &p(p-2) \phi^2 |\nabla T|^{p-3} \langle \nabla |\nabla T| \otimes \nabla T, \Delta T + Rm * T \rangle \\ &\leq 2(p-2) \phi^2 |\nabla T|^{\frac{p-2}{2}} |\nabla |\nabla T|^{\frac{p}{2}}| (|\Delta T| + |Rm| |T|) \\ &\leq \delta \phi^2 |\nabla |\nabla T|^{\frac{p}{2}}|^2 + 4p^2 \phi^2 |\nabla T|^{p-2} (|\Delta T|^2 + |Rm|^2 |T|^2) \end{aligned}$$

Thus we obtain

$$\begin{aligned} \phi^2 |\nabla |\nabla T|^{\frac{p}{2}}|^2 &\leq 2 \operatorname{div}(\phi^2 |\nabla T|^{\frac{p}{2}} \nabla |\nabla T|^{\frac{p}{2}}) - p \operatorname{div}(\phi^2 |\nabla T|^{p-2} \langle \nabla T, \Delta T + Rm * T \rangle) \\ &\quad + C(p^2 \phi^2 |\nabla T|^{p-2} |\Delta T|^2 + p^2 \phi^2 |\nabla T|^{p-2} |Rm|^2 |T|^2 \\ &\quad + p \phi^2 |Rm| |\nabla T|^p + p |\nabla \phi|^2 |\nabla T|^p) \end{aligned}$$

In our case, $Rm = Rm^M + Rm^N(dG, dG)$, since $|\nabla G|^2 |Rm^N|$ is bounded, so we have $p \phi^2 |(\nabla G)^2 * Rm^N| |\nabla T|^p \leq C p \phi^2 |\nabla T|^p$, thus the above inequality holds even if we refer Rm as Rm^M .

Come back to the Sobolev inequality

$$\begin{aligned} \left\{ \int (\phi |\nabla T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int |\nabla \phi|^2 |\nabla T|^p + \int \phi^2 |\nabla |\nabla T|^{\frac{p}{2}}|^2 \right) \\ &\leq C(p^2 \int \phi^2 |\nabla T|^{p-2} |\Delta T|^2 + p^2 \int \phi^2 |\nabla T|^{p-2} |Rm|^2 |T|^2 \\ &\quad + p \int \phi^2 |Rm| |\nabla T|^p + p \int |\nabla \phi|^2 |\nabla T|^p) \end{aligned}$$

By Hölder inequality and our assumption of small integral of curvature, the term

$$p \int \phi^2 |Rm| |\nabla T|^p \leq p \left\{ \int |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \left\{ \int (\phi |\nabla T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}}$$

can be absorbed by the left, thus we get

$$\begin{aligned} \left\{ \int (\phi |\nabla T|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla T|^{p-2} |\Delta T|^2 + \int \phi^2 |\nabla T|^{p-2} |Rm|^2 |T|^2 \right. \\ &\quad \left. + \int |\nabla \phi|^2 |\nabla T|^p \right) \end{aligned}$$

□

Now, we will use lemma 3.2, 3.3 and 3.4 to get some a priori estimation.

Lemma 3.5. There exist a constant $\epsilon = \epsilon(C_S, n)$, if

$$\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \leq \epsilon,$$

then

$$\left\{ \int \phi^2 |\nabla^2 G|^2 \right\}^{\frac{1}{2}} \leq C \left\{ \int |\nabla G|^2 \right\}^{\frac{1}{2}} \quad (3.6)$$

Proof. Apply lemma 3.3 to the equation

$$\Delta \nabla G = Rm^N * (\nabla G)^3 + Ric^M * \nabla G$$

In (3.4), $T = \nabla G, X = 0, Y = Rm^N * (\nabla G)^3 \approx |\nabla G|$, so we obtain the lemma. □

Theorem 3.6. For any $p \geq 2$, there exist a constant $\epsilon = \epsilon(C_S, n)$ such that if

$$\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \leq \frac{\epsilon}{p},$$

then

$$\begin{aligned} \left\{ \int_{B(0, \frac{1}{2})} |\nabla^2 G|^p \right\}^{\frac{1}{p}} &\leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \\ \left\{ \int_{B(0, \frac{1}{2})} |Rm|^p \right\}^{\frac{1}{p}} &\leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \end{aligned}$$

where $C = C(C_S, p, n)$.

Proof. Since the curvature equation is coupled with the harmonic map equation in lemma 2.2, we will see that we have to control the two term in the lemma simultaneously. Recall the Iteration lemma II 3.4, let $T = \nabla G$, then we have

$$\left\{ \int (\phi |\nabla^2 G|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \leq C \left(\int \phi^2 |\nabla^2 G|^{p-2} |\Delta \nabla G|^2 + \int \phi^2 |\nabla^2 G|^{p-2} |Rm|^2 |\nabla G|^2 + \int |\nabla \phi|^2 |\nabla^2 G|^p \right)$$

By the equation

$$\Delta \nabla G = Rm^N * (\nabla G)^3 + Ric^M * \nabla G$$

then

$$|\Delta \nabla G|^2 \leq C(|Rm|^2 + |\nabla G|^2)$$

Therefore

$$\left\{ \int (\phi |\nabla^2 G|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \leq C \left(\int \phi^2 |\nabla^2 G|^{p-2} |Rm|^2 + \int \phi^2 |\nabla^2 G|^{p-2} |\nabla G|^2 + \int |\nabla \phi|^2 |\nabla^2 G|^p \right)$$

Now apply the h older inequality with dual index $\frac{p-2}{p} + \frac{2}{p} = 1$, and we obtain,

$$\left\{ \int (\phi |\nabla^2 G|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \leq C \left(\int \phi^2 |Rm|^p + \int (\phi^2 + |\nabla \phi|^2) |\nabla^2 G|^p + \int \phi^2 |\nabla G|^p \right)$$

Furthermore, taking the p - th root of both side with a trivial inequality

$$\left(\sum_{i=1}^N a_i \right)^{\frac{1}{p}} \leq N^{\frac{1}{p}} \sum_{i=1}^N a_i^{\frac{1}{p}}$$

we get

$$\begin{aligned} \left\{ \int \phi^{2\gamma} |\nabla^2 G|^{p\gamma} \right\}^{\frac{1}{p\gamma}} &\leq C^{\frac{1}{p}} \left(\sup |\nabla \phi|^{\frac{2}{p}} \left\{ \int_{supp \phi} |\nabla^2 G|^p \right\}^{\frac{1}{p}} + \left\{ \int_{supp \phi} |Rm|^p \right\}^{\frac{1}{p}} \right. \\ &\quad \left. + \left\{ \int_{supp \phi} |\nabla G|^p \right\}^{\frac{1}{p}} \right) \end{aligned} \quad (3.7)$$

Therefore, we obtain a priori estimation of $\nabla^2 G$ but involves Rm .

Now we turn to the estimation on Rm . Recall computation in lemma 3.2, replacing T by Rm ,

$$\phi^2 |\nabla |Rm|^{\frac{p}{2}}|^2 \leq 2 \operatorname{div}(\phi^2 |Rm|^{\frac{p}{2}} \nabla |Rm|^{\frac{p}{2}}) - p \phi^2 |Rm|^{p-2} \langle Rm, \Delta Rm \rangle + 4 |\nabla \phi|^2 |Rm|^p$$

Combine with the equation of the curvature

$$\Delta Rm = \nabla^2 Ric + Rm * Rm$$

we have

$$\begin{aligned} -\phi^2 |Rm|^{p-2} \langle Rm, \Delta Rm \rangle &= -\phi^2 |Rm|^{p-2} \langle Rm, \nabla^2 Ric + Rm * Rm \rangle \\ &= -\operatorname{div}(\phi^2 |Rm|^{p-2} \langle Rm, \nabla Ric \rangle) + \phi^2 |Rm|^{p-2} \langle \delta Rm, \nabla Ric \rangle \\ &\quad + \frac{2(p-2)}{p} \phi^2 |Rm|^{\frac{p}{2}-2} \langle Rm, \nabla |Rm|^{\frac{p}{2}} \otimes \nabla Ric \rangle \\ &\quad + 2\phi |Rm|^{p-2} \langle Rm, \nabla \phi \otimes \nabla Ric \rangle - \phi^2 |Rm|^{p-2} \langle Rm, Rm * Rm \rangle \\ &\leq -\operatorname{div}(\phi^2 |Rm|^{p-2} \langle Rm, \nabla Ric \rangle) + \frac{\delta}{p} \phi^2 |\nabla |Rm|^{\frac{p}{2}}|^2 \\ &\quad + C(p \phi^2 |Rm|^{p-2} |\nabla Ric|^2 + |\nabla \phi|^2 |Rm|^p + \phi^2 |Rm|^{p+1}) \end{aligned}$$

and consequently,

$$\begin{aligned} \phi^2 |\nabla |Rm|^{\frac{p}{2}}|^2 &\leq 2 \operatorname{div}(\phi^2 |Rm|^{\frac{p}{2}} \nabla |Rm|^{\frac{p}{2}}) - p \operatorname{div}(\phi^2 |Rm|^{p-2} \langle Rm, \nabla Ric \rangle) \\ &\quad + C(p^2 \phi^2 |Rm|^{p-2} |\nabla Ric|^2 + p \phi^2 |Rm|^{p+1} + p |\nabla \phi|^2 |Rm|^p) \end{aligned}$$

by the Sobolev Inequality, absorbing the term $\int p \phi^2 |Rm|^{p+1}$ by the left, then

$$\left\{ \int (\phi |Rm|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \leq C \left(\int |\nabla \phi|^2 |Rm|^p + \int \phi^2 |Rm|^{p-2} |\nabla Ric|^2 \right)$$

On the other hand, coupled with the Harmonic - Einstein equation (1.3), $Ric = \lambda g + \langle dG, dG \rangle$. Therefore, $\nabla Ric = \nabla^2 G * \nabla G$, and then

$$\begin{aligned} \left\{ \int (\phi |Rm|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int |\nabla \phi|^2 |Rm|^p + \int \phi^2 |Rm|^{p-2} |\nabla^2 G|^2 \right) \\ &\leq C \left(\int (\phi^2 + |\nabla \phi|^2) |Rm|^p + \int \phi^2 |\nabla^2 G|^p \right) \end{aligned}$$

As did for $\nabla^2 G$ (3.7), we have

$$\left\{ \int \phi^{2\gamma} |Rm|^{p\gamma} \right\}^{\frac{1}{p\gamma}} \leq C^{\frac{1}{p}} (\sup |\nabla \phi|^{\frac{2}{p}} \left\{ \int_{\operatorname{supp} \phi} |Rm|^p \right\}^{\frac{1}{p}} + \left\{ \int_{\operatorname{supp} \phi} |\nabla^2 G|^p \right\}^{\frac{1}{p}}) \quad (3.8)$$

By taking $p_i = 2\gamma^i$, $\operatorname{supp} \phi_i \subset B_i := B(0, \frac{1}{2} + (\frac{1}{2})^i)$, $\phi_i \equiv 1$ on B_{i+1} , and $|\nabla \phi_i| \leq 2^{i+2}$. Define

$$\begin{aligned} \Phi_i(\nabla^2 G) &= \left\{ \int_{B_i} |\nabla^2 G|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}} \\ \Psi_i(Rm) &= \left\{ \int_{B_i} |Rm|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}} \end{aligned}$$

With (3.7) and (3.8), we obtain a coupled iteration relation

$$\begin{bmatrix} \Phi_{i+1}(\nabla^2 G) \\ \Psi_{i+1}(Rm) \end{bmatrix} \leq C^{\frac{1}{2\gamma^i}} \begin{bmatrix} 2^{\frac{i}{2\gamma^i}} & 1 \\ 1 & 2^{\frac{i}{2\gamma^i}} \end{bmatrix} \begin{bmatrix} \Phi_i(\nabla^2 G) \\ \Psi_i(Rm) \end{bmatrix} + C^{\frac{1}{2\gamma^i}} \begin{bmatrix} \Phi_i(\nabla G) \\ 0 \end{bmatrix}$$

Denote $\lambda_i = 2^{\frac{i}{2\gamma^i}}$, and use the following fact on matrix:

$$\begin{bmatrix} \lambda & 1 \\ 1 & \lambda \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda+1 & 0 \\ 0 & \lambda-1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T$$

By iterating on i , and consequently,

$$\begin{aligned} &\begin{bmatrix} \Phi_{i+1}(\nabla^2 G) \\ \Psi_{i+1}(Rm) \end{bmatrix} \\ &\leq \frac{1}{2} C^{\sum_{j=0}^i \frac{1}{2\gamma^j}} \begin{bmatrix} \prod_{j=0}^i (\lambda_j + 1) + \prod_{j=0}^i (\lambda_j - 1) & \prod_{j=0}^i (\lambda_j + 1) - \prod_{j=0}^i (\lambda_j - 1) \\ \prod_{j=0}^i (\lambda_j + 1) - \prod_{j=0}^i (\lambda_j - 1) & \prod_{j=0}^i (\lambda_j + 1) + \prod_{j=0}^i (\lambda_j - 1) \end{bmatrix} \begin{bmatrix} \Phi_0(\nabla^2 G) \\ \Psi_0(Rm) \end{bmatrix} \\ &\quad + \sum_{j=0}^i C^{\sum_{k=j}^i \frac{1}{2\gamma^k}} 2^{\sum_{k=j+1}^i \frac{k}{2\gamma^k}} \begin{bmatrix} \Phi_j(\nabla G) \\ 0 \end{bmatrix} \\ &\leq C \begin{bmatrix} e^{c_1(n)i} + c_2(n) & e^{c_1(n)i} - c_2(n) \\ e^{c_1(n)i} - c_2(n) & e^{c_1(n)i} + c_2(n) \end{bmatrix} \begin{bmatrix} \Phi_0(\nabla^2 G) \\ \Psi_0(Rm) \end{bmatrix} + C^i \begin{bmatrix} \Phi_0(\nabla G) \\ 0 \end{bmatrix} \quad (3.9) \end{aligned}$$

where we have used the facts: $\prod_{j=0}^i (2^{\frac{j}{2\gamma^j}} + 1) = e^{\sum_{j=0}^i \ln(2^{\frac{j}{2\gamma^j}} + 1)} \leq e^{\sum_{j=0}^i 2^{\frac{j}{2\gamma^j}}} \leq e^{c_1(n)i}$,
 $\prod_{j=0}^i (2^{\frac{j}{2\gamma^j}} - 1) \leq c_2(n)$, $\sum \frac{1}{q\gamma^j} = \frac{\gamma}{q(\gamma-1)}$, $\sum \frac{i}{q\gamma^j} = \frac{\gamma}{q(\gamma-1)^2}$.

For the initial condition, lemma 3.5 implies

$$\Phi_0(\nabla^2 G) = \left\{ \int_{B_0} |\nabla^2 G|^2 \right\}^{\frac{1}{2}} \leq C \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}}$$

and also as explained in lemma 3.1, the volume of geodesic ball is comparable with Euclidean ball, then

$$\Psi_0(Rm) = \left\{ \int_{B_0} |Rm|^2 \right\}^{\frac{1}{2}} \leq C \left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}}$$

Finally, come back to (3.9), we obtain

$$\begin{aligned} \left\{ \int_{B_i} |\nabla^2 G|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}} &= C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \\ \left\{ \int_{B_i} |Rm|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}} &= C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \end{aligned}$$

where $C = C(n, C_S, i)$. \square

Before we starting to prove the ϵ - regularity 1.3, let us state the Moser iteration lemma.

Lemma 3.7. (Moser iteration [GT83] [BKN89]) Suppose a nonnegative function u satisfies $\Delta u \geq -fu - h - cu$ with $f \in L^q, q > \frac{n}{2}$, $g \in L^{q'}, q' > \frac{n}{2}$, c is some constant, and $u \in L^p$ for some $p \in [p_0, p_1]$ where $p_0 > 1$. Since we do analysis on manifolds, we also assume bounded C_S and Euclidean volume growth, i.e. $\text{vol}(B(0, r)) \leq Vr^n$. Then there exists a constant $C = C(p_0, p_1, C_S, V, c, \|f\|_{L^q})$ so that

$$\sup_{B(0, \frac{r}{2})} |u| \leq Cr^{-\frac{n}{p}} \left\{ \int_{B(0, r)} |u|^p \right\}^{\frac{1}{p}} + Cr^{-\frac{n}{q'}} \left\{ \int_{B(0, r)} |h|^{q'} \right\}^{\frac{1}{q'}} \quad (3.10)$$

Since all the inequalities in the main theorem 1.3 are scale invariant, we may assume $r = 1$ for simplicity, and then theorem 3.9 is equivalent to theorem 1.3.

Theorem 3.8. For any $k \in \mathbb{N}$ and $p \geq 2$, there exist a constant $\epsilon = \epsilon(C_S, n)$ such that if

$$\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \leq \frac{\epsilon}{p},$$

then

$$\left\{ \int_{B(0, \frac{1}{2})} |\nabla^{k+2} G|^2 \right\}^{\frac{1}{2}} \leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \quad (3.11)$$

$$\left\{ \int_{B(0, \frac{1}{2})} |\nabla^{k+2} G|^p \right\}^{\frac{1}{p}} \leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \quad (3.12)$$

$$\left\{ \int_{B(0, \frac{1}{2})} |\nabla^k Rm|^2 \right\}^{\frac{1}{2}} \leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \quad (3.13)$$

$$\left\{ \int_{B(0, \frac{1}{2})} |\nabla^k Rm|^p \right\}^{\frac{1}{p}} \leq C \left(\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right) \quad (3.14)$$

where $C = C(C_S, k, p, n)$.

Theorem 3.9. (ϵ - regularity) There exist a $\epsilon = \epsilon(C_S, n)$ such that if

$$\left\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} \leq \epsilon$$

then for any $k \in \mathbb{N}$, we have

$$\sup_{B(0, \frac{1}{2})} |\nabla^{k+1} G| \leq C(\{\int_{B(0,1)} |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int_{B(0,1)} |\nabla G|^2\}^{\frac{1}{2}}) \quad (3.15)$$

$$\sup_{B(0, \frac{1}{2})} |\nabla^{k-1} Rm| \leq C(\{\int_{B(0,1)} |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int_{B(0,1)} |\nabla G|^2\}^{\frac{1}{2}}) \quad (3.16)$$

where $C = C(C_S, k, n)$.

Remark 2. We will proof theorem 3.8 and theorem 3.9 together by induction on k . Strictly speaking, for bigger k one need shrink the ball further after each step of the iteration. We will take this for granted for short, but note that the constant C will depends on k .

Before we starting the proof, let us present the main idea. As did in the case $k = 0$, see theorem 3.6, if we apply (3.3) on the equation $\Delta \nabla^{k+1} G$, we can get the L^2 estimation on $\nabla^{k+2} G$; Similarly, if we apply (3.3) on the equation $\Delta \nabla^{k-1} Rm$, we can get the L^2 estimation on $\nabla^k Rm$, thus we get (3.11) and (3.13). If $2\gamma > \frac{n}{2}$, namely $n \leq 5$, with Sobolev inequality, (3.11) and (3.13) is enough for Moser iteration to bound the curvature and harmonic map. While for the higher dimension case, we must apply the iteration lemma II 3.4 to the equation $\Delta \nabla^{k+2} G$ and $\Delta \nabla^k Rm$ to improve the integrality order up to $p > \frac{n}{2}$. However, as we have already seen in the case $k = 0$, we can not get a priori estimation for $|\nabla^k Rm|$ and $|\nabla^{k+2} G|$ separately like $n \leq 5$, but get (3.12) and (3.14) simultaneously, since our equation is a coupled system. Once the integrality order is bigger than $p > \frac{n}{2}$, one can apply the Moser iteration lemma 3.7 to get the L^∞ estimate for $|\nabla^{k+1} G|$ and $|\nabla^{k-1} Rm|$, therefore we get (3.15) and (3.16).

Proof. We have already proved the case $k = 0$ in theorem 3.6. Moreover, we will see the case $k = 1$ in theorem 3.8 does not require theorem 3.9. The theorem 3.9 will begin from $k = 1$ and the case $k = 1$ will be proved in step III, which require the case $k = 1$ in theorem 3.8. Thus the induction process is well ordered. Now we assume all the inequalities in theorem 3.9 and 3.8 hold for the case from $k = 0$ through out to $k - 1$.

Step I: Recall lemma 2.2, for the Harmonic - Einstein equation (1.3), we have the coupled system (2.3) and (2.4) for the full curvature tensor Rm and harmonic map G . Now if we apply lemma 3.3 on the equation (2.4)

$$\begin{aligned} \Delta \nabla^{k+1} G &= \sum_{i=0}^k \nabla^i Rm * \nabla^{k-i+1} G + \sum_{p=3}^{k+3} \sum_{i_1+\dots+i_p=k-p+3} \nabla^{i_1+1} G * \dots * \nabla^{i_p+1} G \\ &= \nabla(\nabla^{k-1} Rm * \nabla G) + Rm * \nabla^{k+1} G + \sum_{i=1}^{k-1} \nabla^i Rm * \nabla^{k-i+1} G \\ &\quad + \nabla G * \nabla G * \nabla^{k+1} G + \sum_{p=3}^{k+3} \sum_{i_1+\dots+i_p=k-p+3, i_* < k} \nabla^{i_1+1} G * \dots * \nabla^{i_p+1} G \end{aligned}$$

with $T = \nabla^{k+1} G$, $c = |\nabla G|^2$, $X = \nabla^{k-1} Rm * \nabla G$. When $k = 1$, then

$$\Delta \nabla^2 G = \nabla(Rm * \nabla G) + Rm * \nabla^2 G + (\nabla G)^2 * \nabla^2 G + (\nabla G)^4,$$

Therefore, $Y = (\nabla G)^4$ in our notation, and $|Y| \leq C\{\int_{B(0,1)} |\nabla G|^2\}^{\frac{1}{2}}$ without the induction in theorem 3.9; When $k \geq 2$, by induction, (3.15) and (3.16) hold up to $k - 1$, namely, $|\nabla^j Rm|$ and $|\nabla^{j+2} G|$ are bounded for $j \leq k - 2$, then

$$\begin{aligned} |Y| &= \left| \sum_{i=1}^{k-1} \nabla^i Rm * \nabla^{k-i+1} G + \sum_{p=3}^{k+3} \sum_{i_1+\dots+i_p=k-p+3, i_* < k} \nabla^{i_1+1} G * \dots * \nabla^{i_p+1} G \right| \\ &\leq C|\nabla^{k-1} Rm| + C(\{\int_{B(0,1)} |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int_{B(0,1)} |\nabla G|^2\}^{\frac{1}{2}}) \end{aligned}$$

Come back to lemma 3.3, we have

$$\begin{aligned}
\int \phi^2 |\nabla^{k+2} G|^2 &\leq C \left(\int \phi^2 |X|^2 + \int (\phi^2 + |\nabla \phi|^2) |T|^2 + \int \phi^2 |Y|^2 \right) \\
&\leq C \left(\int \phi^2 |\nabla^{k-1} Rm|^2 + \int (\phi^2 + |\nabla \phi|^2) |\nabla^{k+1} G|^2 \right. \\
&\quad \left. + (\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \}^{\frac{2}{n}} + \{ \int_{B(0,1)} |\nabla G|^2 \}^{\frac{1}{2}})^2 \right)
\end{aligned}$$

By induction, (3.11) and (3.13) hold for with $k-1$, so (3.11) holds for k .

Similarly, if we apply (3.3) on the equation 2.3,

$$\Delta \nabla^{k-1} Rm = Rm * \nabla^{k-1} Rm + \sum_{i=0}^{k+1} \nabla^{i+1} G * \nabla^{k+2-i} G + \sum_{i=1}^{k-2} \nabla^i Rm * \nabla^{k-i-1} Rm$$

with $T = \nabla^{k-1} Rm$, $c = 0$, $X = 0$, and

$$\begin{aligned}
|Y| &= |\nabla G * \nabla^{k+2} G + \sum_{i=1}^k \nabla^{i+1} G * \nabla^{k+2-i} G + \sum_{i=1}^{k-2} \nabla^i Rm * \nabla^{k-i-1} Rm| \\
&\leq (|\nabla^{k+2} G| + \sum_{i=1}^k |\nabla^{i+1} G|^2 + \sum_{i=1}^{k-2} |\nabla^i Rm|^2)
\end{aligned}$$

then we have

$$\begin{aligned}
\int \phi^2 |\nabla^k Rm|^2 &\leq C \left(\int (\phi^2 + |\nabla \phi|^2) |T|^2 + \int \phi^2 |Y|^2 \right) \\
&= C \left(\int (\phi^2 + |\nabla \phi|^2) |\nabla^{k-1} Rm|^2 + \int \phi^2 |\nabla^{k+2} G|^2 \right. \\
&\quad \left. + \sum_{i=1}^k \int \phi^2 |\nabla^{i+1} G|^4 + \sum_{i=1}^{k-2} \int \phi^2 |\nabla^i Rm|^4 \right)
\end{aligned}$$

By induction, (3.11) - (3.14) hold up to $k-1$, and (3.11) holds for k which is proved just now, then we have (3.13) for k .

Step II, apply the iteration lemma II 3.4 to the equation (2.4), with $T = \nabla^{k+1} G$, then we have

$$\begin{aligned}
\{ \int (\phi |\nabla^{k+2} G|^{\frac{n}{2}})^{2\gamma} \}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla^{k+2} G|^{p-2} |\Delta \nabla^{k+1} G|^2 \right. \\
&\quad \left. + \int \phi^2 |\nabla^{k+2} G|^{p-2} |Rm|^2 |\nabla^{k+1} G|^2 + \int |\nabla \phi|^2 |\nabla^{k+2} G|^p \right)
\end{aligned}$$

Applying Schwartz inequality to the equation (2.4), $\Delta \nabla^{k+1} G$: when $k = 1$, we have

$$|\Delta \nabla^2 G|^2 \leq C (|Rm|^2 |\nabla^2 G|^2 + |\nabla Rm|^2 + |\nabla^2 G|^2 + \int_{B(0,1)} |\nabla G|^2)$$

therefore, we do not use the induction in theorem 3.9; when $k \geq 2$, by induction, (3.15) and (3.16) hold up to $k-1$, then we have

$$\begin{aligned}
|\Delta \nabla^{k+1} G|^2 &\leq C (|Rm|^2 |\nabla^{k+1} G|^2 + |\nabla^k Rm|^2 + |\nabla^{k-1} Rm|^2 + |\nabla^{k+1} G|^2 \\
&\quad + (\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \}^{\frac{2}{n}} + \{ \int_{B(0,1)} |\nabla G|^2 \}^{\frac{1}{2}})^2)
\end{aligned}$$

Replacing the Laplacian term in the above integral inequality,

$$\begin{aligned} \left\{ \int (\phi |\nabla^{k+2} G|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla^{k+2} G|^{p-2} |Rm|^2 |\nabla^{k+1} G|^2 + \int \phi^2 |\nabla^{k+2} G|^{p-2} |\nabla^k Rm|^2 \right. \\ &\quad + \int \phi^2 |\nabla^{k+2} G|^{p-2} (|\nabla^{k-1} Rm|^2 + |\nabla^{k+1} G|^2) + \int |\nabla \phi|^2 |\nabla^{k+2} G|^p \\ &\quad \left. + \int \phi^2 |\nabla^{k+2} G|^{p-2} (\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \}^{\frac{2}{n}} + \{ \int_{B(0,1)} |\nabla G|^2 \}^{\frac{1}{2}})^2 \right) \end{aligned}$$

By Hölder inequality with $\frac{p-2}{p} + \frac{2}{p} = 1$, we have

$$\begin{aligned} \left\{ \int (\phi |\nabla^{k+2} G|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int (\phi^2 + |\nabla \phi|^2) |\nabla^{k+2} G|^p + \int \phi^2 |\nabla^k Rm|^p + \int \phi^2 |\nabla^{k-1} Rm|^p \right. \\ &\quad + \int \phi^2 |\nabla^{k+1} G|^p + \int \phi^2 |\nabla^{k+1} G|^{2p} + \int \phi^2 |Rm|^{2p} \\ &\quad \left. + (\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \}^{\frac{2}{n}} + \{ \int_{B(0,1)} |\nabla G|^2 \}^{\frac{1}{2}})^p \right) \\ &\leq C \left(\int (\phi^2 + |\nabla \phi|^2) |\nabla^{k+2} G|^p + \int \phi^2 |\nabla^k Rm|^p \right. \\ &\quad \left. + (\{ \int_{B(0,1)} |Rm|^{\frac{n}{2}} \}^{\frac{2}{n}} + \{ \int_{B(0,1)} |\nabla G|^2 \}^{\frac{1}{2}})^p \right) \end{aligned} \quad (3.17)$$

where the last inequality follows from the induction, namely, (3.11) - (3.14) for $k-1$. Therefore we get a priori estimation on $|\nabla^{k+2} G|$ but involves $\nabla^k Rm$.

Now we turn to the estimation of $\nabla^k Rm$. By the Iteration lemma II 3.4 again to (2.3), let $T = \nabla^{k-1} Rm$,

$$\begin{aligned} \left\{ \int (\phi |\nabla^k Rm|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla^k Rm|^{p-2} |\Delta \nabla^{k-1} Rm|^2 \right. \\ &\quad \left. + \int \phi^2 |\nabla^k Rm|^{p-2} |Rm|^2 |\nabla^{k-1} Rm|^2 + \int |\nabla \phi|^2 |\nabla^k Rm|^p \right) \end{aligned}$$

Applying Schwartz inequality to (2.3),

$$|\Delta \nabla^{k-1} Rm|^2 \leq C (|\nabla^{k+2} G|^2 + \sum_{i=1}^k |\nabla^{i+1} G|^2 |\nabla^{k+2-i} G|^2 + \sum_{i=0}^{k-1} |\nabla^i Rm|^2 |\nabla^{k-i-1} Rm|^2)$$

Then we have

$$\begin{aligned} &\left\{ \int (\phi |\nabla^k Rm|^{\frac{p}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} \\ &\leq C \left(\int \phi^2 |\nabla^k Rm|^{p-2} |\nabla^{k+2} G|^2 + \sum_{i=1}^k \int \phi^2 |\nabla^k Rm|^{p-2} |\nabla^{i+1} G|^2 |\nabla^{k+2-i} G|^2 \right. \\ &\quad \left. + \sum_{i=0}^{k-1} \int \phi^2 |\nabla^k Rm|^{p-2} |\nabla^i Rm|^2 |\nabla^{k-i-1} Rm|^2 + \int |\nabla \phi|^2 |\nabla^k Rm|^p \right) \end{aligned}$$

By Hölder inequality with $\frac{p-2}{p} + \frac{2}{p} = 1$,

$$\begin{aligned}
\left\{ \int (\phi |\nabla^k Rm|^{\frac{n}{2}})^{2\gamma} \right\}^{\frac{1}{\gamma}} &\leq C \left(\int \phi^2 |\nabla^{k+2} G|^p + \int (\phi^2 + |\nabla \phi|^2) |\nabla^k Rm|^p \right. \\
&\quad \left. + \sum_{i=1}^k \int \phi^2 |\nabla^{i+1} G|^{2p} + \sum_{i=0}^{k-1} \int \phi^2 |\nabla^i Rm|^{2p} \right) \\
&\leq C \left(\int \phi^2 |\nabla^{k+2} G|^p + \int (\phi^2 + |\nabla \phi|^2) |\nabla^k Rm|^p \right. \\
&\quad \left. + \left(\int_{B(0,1)} |Rm|^{\frac{n}{2}} \right)^{\frac{2}{n}} + \left\{ \int_{B(0,1)} |\nabla G|^2 \right\}^{\frac{1}{2}} \right)^p \quad (3.18)
\end{aligned}$$

As did in theorem 3.6, see (3.9), taking $p_i = 2\gamma^i$, $\text{supp } \phi_i \subset B_i := B(0, \frac{1}{2} + (\frac{1}{2})^i)$ and $\phi_i \equiv 1$ on B_{i+1} , and $|\nabla \phi| \leq 2^{i+2}$. Define

$$\begin{aligned}
\Phi_i(\nabla^{k+2} G) &= \left\{ \int_{B_i} |\nabla^{k+2} G|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}} \\
\Psi_i(\nabla^k Rm) &= \left\{ \int_{B_i} |\nabla^k Rm|^{2\gamma^i} \right\}^{\frac{1}{2\gamma^i}}
\end{aligned}$$

and

$$C_0 = C \left(\left\{ \int |Rm|^{\frac{n}{2}} \right\}^{\frac{2}{n}} + \left\{ \int |\nabla G|^2 \right\}^{\frac{1}{2}} \right)$$

With (3.17) and (3.18), we obtain a coupled iteration sequence

$$\begin{bmatrix} \Phi_{i+1}(\nabla^{k+2} G) \\ \Psi_{i+1}(\nabla^k Rm) \end{bmatrix} \leq C^{\frac{1}{2\gamma^i}} \begin{bmatrix} 2^{\frac{i}{2\gamma^i}} & 1 \\ 1 & 2^{\frac{i}{2\gamma^i}} \end{bmatrix} \begin{bmatrix} \Phi_i(\nabla^{k+2} G) \\ \Psi_i(\nabla^k Rm) \end{bmatrix} + C^{\frac{1}{2\gamma^i}} \begin{bmatrix} C_0 \\ C_0 \end{bmatrix}$$

With the same iteration process as in (3.9), we obtain

$$\begin{bmatrix} \Phi_{i+1}(\nabla^{k+2} G) \\ \Psi_{i+1}(\nabla^k Rm) \end{bmatrix} \leq C \begin{bmatrix} e^{c_1(n)i} + c_2(n) & e^{c_1(n)i} - c_2(n) \\ e^{c_1(n)i} - c_2(n) & e^{c_1(n)i} + c_2(n) \end{bmatrix} \begin{bmatrix} \Phi_0(\nabla^{k+2} G) \\ \Psi_0(\nabla^k Rm) \end{bmatrix} + C_i \begin{bmatrix} C_0 \\ C_0 \end{bmatrix}$$

The initial condition (3.11) and (3.13) for $i = 0$, are proved in step I, and therefore we proved (3.12) and (3.14) for the case k .

Step III: We will apply Moser iteration to get the L^∞ estimation (3.15) and (3.16).

For initial case $k = 1$, once we have $L^q(q > \frac{n}{2})$ bound of $\nabla^2 Ric$ and $L^{q'}(q' > \frac{n}{2})$ bound of Rm , by lemma 3.7, we can apply the Moser iteration to the equation

$$-\Delta |Rm| \leq C(n) |\nabla^2 Ric| + C(n) |Rm| |Rm|$$

to obtain the L^∞ estimation of the full curvature tensor. Note that in the proof of step I and II for the case $k = 1$, we do not need the induction in theorem 3.9, therefore we have (3.11) - (3.14) hold for $k = 1$. If we take $p = 2^i > n$, namely, $i = \lfloor \frac{\ln n}{\ln 2} \rfloor + 1$ in theorem 3.8, then we have $L^{\frac{p}{2}}$ bound of $\sum_{j=1}^3 |\nabla^j G|^2 \geq |\nabla^2 Ric|$ and L^p bound of $|Rm|$, which implies the L^∞ bound of the full curvature tensor. On the other hand, we also have $L^p(p > n)$ bound of ∇Rm , the same argument on the equation

$$-\Delta |\nabla^2 G| \leq C(n) (|Rm| |\nabla^2 G| + |\nabla G|^2 |\nabla^2 G| + |\nabla G| |\nabla Rm|^2 + |\nabla G|^4)$$

will give the L^∞ estimation on the derivation of G up to second order.

For any $k > 1$, we have assumed, by induction, (3.15) and (3.16) hold up to $k - 1$.

Apply Moser iteration lemma 3.7 to the equation,

$$\begin{aligned} -\Delta|\nabla^{k+1}G| &\leq C(n)(|Rm||\nabla^{k+1}G| + |\nabla G|^2|\nabla^{k+1}G| + |\nabla G||\nabla^k Rm| + |\nabla^2 G||\nabla^{k-1}Rm|) \\ &\quad + C(\{\int |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int |\nabla G|^2\}^{\frac{1}{2}}) \end{aligned}$$

we have $L^p(p > \frac{n}{2})$ norm of $|\nabla G||\nabla^k Rm| + |\nabla^2 G||\nabla^{k-1}Rm| + C(\{\int |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int |\nabla G|^2\}^{\frac{1}{2}})$, and also L^p norm of $|\nabla^{k+1}G|$ by (3.12) - (3.14). Apply lemma 3.7 once more, we obtain the L^∞ estimation of $|\nabla^{k+1}G|$.

Similarly, on the equation

$$\Delta\nabla^{k-1}Rm = Rm * \nabla^{k-1}Rm + \sum_{i=0}^{k+1} \nabla^{i+1}G * \nabla^{k+2-i}G + \sum_{i=1}^{k-2} \nabla^i Rm * \nabla^{k-i-1}Rm$$

by induction on (3.15) and (3.16), therefore

$$\begin{aligned} -\Delta|\nabla^{k-1}Rm| &\leq C(n)(|Rm||\nabla^{k-1}Rm| + |\nabla G||\nabla^{k+2}G| + |\nabla^2 G||\nabla^{k+1}G|) \\ &\quad + C(\{\int |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int |\nabla G|^2\}^{\frac{1}{2}}) \end{aligned}$$

we have $L^{\frac{p}{2}}(p > n)$ bound of $|\nabla^2 G||\nabla^{k+1}G| + C(\{\int |Rm|^{\frac{n}{2}}\}^{\frac{2}{n}} + \{\int |\nabla G|^2\}^{\frac{1}{2}})$, and L^p bound of $\nabla^k Rm$ by (3.12) and (3.14). Apply lemma 3.7 again, we obtain the L^∞ estimation of $\nabla^k Rm$. \square

4 Compactness of Harmonic - Einstein Equation

In this section, we will give a sketch proof on the theorem 1.4, since the argument is very similar to the case of Einstein metrics, [Anderson89], [BKN89], [Tian90], Bach flat metric with constant scalar curvature [TV05b], [AAJV11], Kähler Ricci soliton [CS07], and extremal Kähler metric [CW11]. By the way, the author had also written a detailed proof for the removable singularity theorem in the case of Bach flat metric with constant scalar curvature before this work. As stated in the theorem, we have two aspects to show: one is the convergence of Harmonic - Einstein equation in certain topology, the other is smooth extension of the Harmonic - Einstein equation across the singularity, which is called to be the removable singularity theory.

First, with the assumption in theorem 1.4, we can bound the energy and Sobolev constant, which is appeared in the ϵ - regularity theorem 1.3.

Lemma 4.1. With the assumption in theorem 1.4, there are constants $\Lambda_k, k = 1, 2, 3$, which are depending on X, D, V, E , but not on i , such that

$$\int_{M_i} |Rm(g_i)|^2 \leq \Lambda_1, \int_{M_i} |dG_i|^2 \leq \Lambda_2, C_S(M_i) \leq \Lambda_3.$$

Proof. In fact, C.Croke [Croke80] proved that the isoperimetric constant is bounded above by a constant depending only on a lower bound for the Ricci curvature, lower bound on volume and an upper bound on the diameter. In the later, based on Gromov's technique, Anderson [Anderson92] give a local version, which require on local (Euclidean) volume growth condition. On the other hand, isoperimetric constant is equal to Sobolev constant by Federer - Fleming's theory. In our case, $Ric = \lambda g + \langle dG, dG \rangle \geq \lambda g$, $Diam \leq D$ and $Vol \geq V$, so we have a uniform upper bound for the Sobolev constant: $C_S \leq C(D, V)$.

With Sobolev constant, from lemma 3.1, we have $\sup_{M_i} |\nabla G_i| \leq C(D, V, E)$. Moreover, Ricci curvature is two sided bounded: $|Ric(g_i)| \leq C(D, V, E)$, and the scalar curvature

$R = \lambda n + |dG|^2$ is bounded too. Recall the Gauss - Bonnet formula on compact four manifold M , see [Besse87],

$$\mathcal{X}(M) = \frac{1}{8\pi^2} \int_M |Rm|^2 - |Ric - \frac{1}{4}Rg|^2$$

then we have $\int_{M_i} |Rm|^2 \leq C(X, D, V, E)$. \square

Now, let us give a sketch proof on the theorem 1.4.

Proof. Step I: As in the case of Einstein metric or Bach flat metric with constant curvature, since we have established the local regularity of Harmonic - Einstein equation, then the sequence will converge as stated in the theorem by applying the Cheeger - Gromov convergence, not only the convergence of the metric g_i , but also with function G_i . More precisely, taking $\epsilon = \epsilon(n, C_S)$ in theorem 1.3, consider the sets

$$\mathcal{R}_i(r) = \{x \in M_i | \{\int_{B(x,r)} |Rm|^{\frac{n}{2}}\}^{\frac{n}{2}} < \epsilon\}, \quad \mathcal{S}_i(r) = \{x \in M_i | \{\int_{B(x,r)} |Rm|^{\frac{n}{2}}\}^{\frac{n}{2}} \geq \epsilon\}$$

then $M_i = \mathcal{R}_i(r) \cup \mathcal{S}_i(r)$ and also $\mathcal{R}_i(r_1) \subset \mathcal{R}_i(r_2)$, $\mathcal{S}_i(r_1) \supset \mathcal{S}_i(r_2)$, for any $r_1 > r_2$.

For all $x \in \mathcal{R}_i(r)$, by ϵ - regularity theorem 1.3, for all $k \in \mathbb{N}$, we have

$$\sup_{B(0, \frac{r}{2})} |\nabla^k G_i| \leq \frac{C}{r^k}, \quad \sup_{B(0, \frac{r}{2})} |\nabla^k Rm(g_i)| \leq \frac{C}{r^{k+2}},$$

where $C = C(n, k, \Lambda_1, \Lambda_2, \Lambda_3)$.

Letting $\{B(x_k^i, \frac{r}{4})\}, k \in \mathbb{N}$ be a collection of a maximal family of disjoint geodesic balls in M_i , then $M_i \subset \cup_k B(x_k^i, r)$. There is a uniform bound, independent of i , on the number of points $\{x_k^i \in \mathcal{S}_i(r)\}$, which follows from

$$m \leq \sum_{i=1}^m \epsilon^{-\frac{n}{2}} \int_{B(x_k^i, r)} |Rm|^{\frac{n}{2}} \leq C \epsilon^{-\frac{n}{2}} \int_{M_i} |Rm|^{\frac{n}{2}}, \quad (4.1)$$

where $C = \sup_{x \in M_i} \frac{Vol(B(x, \frac{5r}{4}))}{Vol(B(x, \frac{r}{4}))} \leq C(n, \Lambda_2)$. Without loss generality, we will assume m is fixed, which is independent on i and r .

On the other hand, the uniform Sobolev constant implies uniform noncollapsing, namely, $Vol(B(x, r)) \geq C(C_S)r^n$. Combine the uniform bound of curvature, we have a uniform lower bound on the local injective radius, i.e. $\text{inj}(x) \geq Cr$, $x \in R_i(r)$, see [CGT82]. According Cheeger - Gromov convergence theory [GW88], we can extract a subsequence, so that $(\mathcal{R}_j(r), g_j, G_j)$ converges smoothly to a smooth open Riemannian manifold $(\mathcal{R}_\infty(r), g_\infty, G_\infty)$. Since the convergence is in the $C^\infty(\mathcal{R}_\infty(r))$ topology, then the limit (g_∞, G_∞) still satisfies the Harmonic - Einstein equation on $R_\infty(r)$.

We now choosing a sequence $\{r_k\} \rightarrow 0$ and repeat the above construction by choosing subsequence, we still denote $\{j\}$. Since $\mathcal{R}_i(r_k) \subset \mathcal{R}_i(r_{k+1})$, then we have a sequence of limit spaces with natural inclusions

$$\mathcal{R}_\infty(r_k) \subset \mathcal{R}_\infty(r_{k+1}) \subset \cdots \subset \mathcal{R}_\infty := \text{dir. lim } \mathcal{R}_\infty(r_k)$$

Due to finite capacity of \mathcal{S}_j in (4.1), following the argument of [Anderson89], [Tian90], one can add finite points $\mathcal{S}_\infty = \{b_1, \dots, b_m\}$ to \mathcal{R}_∞ such that $M_\infty := \mathcal{R}_\infty \cup \mathcal{S}_\infty$ is complete with respect g_∞ . Since $|\nabla G|$ is uniformly bounded, $G_\infty(b_k) := \lim_{b \rightarrow b_k} G_\infty(b) \in (N, h)$ is well defined for $k = 1, \dots, m$.

Moreover, with the local regularity, the curvature may blow up at the singularity, but at worst, at a rate of quadratic, i.e. $\sup_{\{x: d(x, \mathcal{S})=r\}} |Rm| \leq \frac{o(r)}{r^2}$, then we know the singularity has a C^0 orbifold structure, see [Tian90] or [TV05a]. Since the energy is concentrated at the

singular set \mathcal{S} , then both $\int_{M_\infty} |Rm|^2$ and $\int_{M_\infty} |dG|^2$ remains bounded, which follows from the lower semi - continuously of energy.

Step II: We will extend the Harmonic - Einstein equation across the singularity. On the limit space with finite singularity, we may assume the finite group is trivial, i.e. $\Gamma_k = \{e\}$, by going to the universal covering space. If the full curvature is uniformly bounded near the singularity, one can construct “good” coordinate, namely, $C^{1,\alpha}$ harmonic coordinate around the singularity [KD81], [BKN89], [Tian90]. And consequently, one can go back to the equation and bootstrapping to improve the regularity. Thus the main task is to bound the full curvature tensor.

We want to get the similar estimation via the Moser iteration on the Riemannian orbifold. However, due to singularity of the manifold structure, this would appear impossible, as we do not know that our elliptic inequality $\Delta u \geq -fu - h$ holds weakly across the singularity. We will easily see the Sobolev inequality does hold despite the singularity, while integration by parts leaves an uncontrollable term $\int |\nabla \phi|^2 u^p$ near the singular point. If $u \in L^p, p > \gamma$, let ϕ be zero on $B(p, r)$ near the singularity, then

$$\int |\nabla \phi|^2 u^p \leq \left\{ \int |\nabla \phi|^n \right\}^{\frac{n}{2}} \left\{ \int_{\text{supp}|\nabla \phi|} |u|^{p\gamma} \right\}^{\frac{1}{\gamma}}$$

become negligible since $\left\{ \int_{B(p,r)} |\nabla \phi|^n \right\}^{\frac{n}{2}}$ is uniformly bounded, this is Siber’s lemma [Sibner85], which is used by [CS07] and [CW11].

When $n = 4$, then $\gamma = 2$; but we only have $u = |Rm| \in L^2$, thus the equation does not hold in the weak sense across the singularity. One approach to overcome this problem is using the Yang - Mill like argument under Hodge gauge to improve the estimation, which is created by Uhlenbeck [Uhlenbeck82a], and developed by Tian [Tian90], [TV05a]. More precisely, by choosing Hodge gauge, integration on the annuls around the singularity point, one can compare the energy of Rm , $f(r) := \int_{B(p_k,r)} |Rm|^2$, with its derivatives, $f'(r) = \int_{S(p_k,r)} |Rm|^2$ to get a differential inequality on $f(r)$. And consequently, one can improve the decay order of $f(r)$.

Lemma 4.2. [Tian90] There are constants ϵ and C such that any connection A on the trivial bundle over a punctured ball $B(0, 1) \setminus \{0\}$ with $\|R_A\| \leq \frac{\epsilon(r)}{r^2}$, is gauge equivalent to a connection A^τ on the annulus $\Omega(r, R) := B(0, R) \setminus \overline{B(0, r)}$ with

1. $d^* A^\tau(r, R) = 0$ in $\overline{\Omega(r, R)}$,
2. $d_\psi^* A^\tau = 0$ on $S(r, R) := \partial\Omega(r, R)$,
3. $\int_{\Omega(r, R)} A(\nabla r) = 0$,
4. $\int_{\Omega(r, 2r)} |A|^2 \leq Cr^2 \int_{\Omega(r, 2r)} |R_A|^2$

where d^* and d_ψ^* are the adjoint operators of the exterior differentials on $\Omega(r, R)$, $S(r, R)$ respectively. Moreover, for suitable constants ϵ and C , the connection A^τ is uniquely determined, up to the transformation $A^\tau \rightarrow u_0 A^\tau u_0^{-1}$ for constant gauge u_0 .¹

Now we will improve the decay order of $\int |Rm|^2$ by the same argument in [Tian90] or [TV05a], but change only a few words, namely, the Ricci term is related to the harmonic map G , see (4.2).

Lemma 4.3. For any $b_k \in \mathcal{S}_\infty$, denote $B(r) := B(b_k, r)$ and $S(r) := \partial B(b_k, r)$. There exists $1 < \beta < 2$ such that for r sufficiently small, we have

$$\sup_{S(r)} |Rm| \leq Cr^{-(2-\beta)}.$$

¹It follows from the uniqueness of Hodge gauge on sphere, Theorem 2.5 [Uhlenbeck82a].

Proof. Choose $r_0 = r$ small, and let us denote $r_i = \frac{1}{2}r_{i-1}$. Let A_i be the connection on $\Omega_i = \Omega(r_i, r_{i-1})$ from lemma 4.2, then there exist Hodge Gauge

$$\begin{aligned} d_\psi^* A_{i\psi} |_{\partial\Omega_i} &= 0 \\ d_\psi^* A_{(i+1)\psi} |_{\partial\Omega_{i+1}} &= 0 \end{aligned}$$

so the restriction $A_{i\psi}$ and $A_{(i+1)\psi}$ differ by a constant gauge on $S(r_i)$ and we may therefore assume that

$$A_{i\psi} |_{S(r_i)} = A_{(i+1)\psi} |_{S(r_i)}$$

and then the curvature is continuous across the S_i , i.e. $(R_{A_i})_{r\psi} = (R_{A_{i+1}})_{r\psi}$ follows from the gauge transformation rule of curvature. Then we compute the L^2 of curvature

$$\begin{aligned} \int_{\Omega_i} |Rm|^2 &= \int_{\Omega_i} \langle D_i A_i - [A_i, A_i], R_{A_i} \rangle \\ &= - \int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle - \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle \\ &\quad - \int_{S_i} \langle (A_i)_\psi, (R_{A_i})_{r\psi} \rangle - \int_{S_{i+1}} \langle (A_i)_\psi, (R_{A_i})_{r\psi} \rangle \end{aligned}$$

Next we sum over i , the boundary terms cancel, except for S_0 and the inner budgetary terms become negligible as $i \rightarrow \infty$,

$$\begin{aligned} \int_{B(r)} |Rm|^2 &= \sum_{i=1}^{\infty} \int_{\Omega_i} |R_{A_i}|^2 \\ &= - \sum_{i=1}^{\infty} \int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle - \sum_{i=1}^{\infty} \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle \\ &\quad + \int_{S(r)} \langle (A_1)_\psi, (R_{A_1})_{r\psi} \rangle \end{aligned}$$

Let us estimate the three term on the right separatively. In fact, on the round sphere S^3 , the first eigenvalue for the Laplacian on coclosed 1 form is 4^2 , then

$$\begin{aligned} \{4 \int_{S^3} |A|^2\}^{\frac{1}{2}} &\leq \{ \int_{S^3} |dA|^2 \}^{\frac{1}{2}} \\ &\leq \{ \int_{S^3} |R_A|^2 \}^{\frac{1}{2}} + C |R_A|_{L^\infty} \{ \int_{S^3} |A|^2 \}^{\frac{1}{2}} \end{aligned}$$

Since the singularity is C^0 orbifold, then the geodesic sphere is convergence to the round sphere after scaling. Therefore we may find monotone function $\epsilon'(r)$ with $\lim_{r \rightarrow 0} \epsilon'(r) = 0$ such that

$$\begin{aligned} \int_{S(r)} \langle (A_1)_\psi, (R_{A_1})_{r\psi} \rangle &\leq \left(\int_{S(r)} |(A_1)_\psi|^2 \right)^{\frac{1}{2}} \left(\int_{S(r)} |(R_{A_1})_{r\psi}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2 - \epsilon'(r)} r \left(\int_{S(r)} |(R_{A_1})_{\psi\psi}|^2 \right)^{\frac{1}{2}} \left(\int_{S(r)} |(R_{A_1})_{r\psi}|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{1}{2} \frac{1}{2 - \epsilon'(r)} r \int_{S(r)} |Rm|^2 \end{aligned}$$

Note that $D_i^* R_{A_i} = \nabla^* Rm = d^\nabla Ric$, and we estimate the first term as

$$\begin{aligned} \int_{\Omega_i} \langle A_i, D_i^* R_{A_i} \rangle &\leq \frac{\delta}{C} r_i^{-2} \int_{\Omega_i} |A_i|^2 + C \delta^{-1} r_i^2 \int_{\Omega_i} |\nabla Ric|^2 \\ &\leq \delta \int_{\Omega_i} |R_{A_1}|^2 + C \delta^{-1} r_i^2 \int_{\Omega_i} |\nabla Ric|^2 \end{aligned}$$

²See also the Corollary 2.6 of [Uhlenbeck82a].

where the C is the constant in 4 - th item of lemma 4.2. Moreover, we have

$$\begin{aligned}
\left| \sum_{i=1}^{\infty} \int_{\Omega_i} \langle [A_i, A_i], R_{A_i} \rangle \right| &\leq \sum_{i=1}^{\infty} C \sup_{\Omega_i} |R_{A_i}| \int_{\Omega_i} |A_i|^2 \\
&\leq \sum_{i=1}^{\infty} C \epsilon(r_{i-1}) \int_{\Omega_i} |R_{A_i}|^2 \\
&= C \epsilon(r) \int_{B(p,r)} |Rm|^2
\end{aligned}$$

For the *Ricci* term, we will prove later that we still have (3.5),

$$\int_{\Omega_i} |\nabla Ric|^2 \leq C r_i^2 \quad (4.2)$$

Combining the above stimulation, we obtain

$$(1 - C\epsilon(r) - \delta) \int_{B(r_0)} |Rm|^2 \leq \frac{1}{2} \frac{1}{2 - \epsilon'(r)} \int_{S(r)} |Rm|^2 + \sum_{i=1}^{\infty} C \delta^{-1} r_i^4$$

Therefore for all r sufficiently small, choosing δ sufficiently small, there exists a small constant $\delta' \geq 0$,

$$\int_{B(p,r)} |Rm|^2 \leq \frac{r}{4 - \delta'} \int_{S(r)} |Rm|^2 + C r^4$$

We denote $f(r) = \int_{B(p,r)} |Rm|^2$, then

$$f(r) \leq \frac{1}{c_1} r f'(r) + c_2 r^4$$

where $c_1 = 4 - \delta'$. Then

$$(r^{-c_1} f(r))' = r^{-c_1} f'(r) - c_1 r^{-c_1-1} f(r) \geq -c_1 c_2 r^{-1+\delta'},$$

Therefore

$$r_0^{-c_1} f(r_0) - r^{-c_1} f(r) = \int_r^{r_0} (r^{-c_1} f(r))' \geq -c_1 c_2 \int_r^{r_0} r^{-1+\delta'}$$

If $\delta' > 0$, then $f(r) \leq C r^{4-\delta'} + \frac{C}{\delta'} r^4 \leq C r^{4-\delta'}$; If $\delta' = 0$, then $f(r) \leq C r^4 + C r^4 \ln r$. In any case, δ' is small, so we can choose $1 < \beta < 2$ such that $f(r) \leq C r^{2\beta}$. Therefore, working on the ball $B(x, \frac{r}{2}) \subset B(p, 1) \setminus \{p\}$, $x \in S(r)$, from the ϵ -regularity theorem 1.3 for the smooth case, we have

$$|Rm|(x) \leq \sup_{B(x, \frac{r}{4})} |Rm| \leq C(n) r^{-2} \{C_S^n \int_{B(x, \frac{r}{2})} |Rm|^2\}^{\frac{1}{2}} \leq C r^{-2+\beta}.$$

□

Proof. (proof of (4.2)) Define $\phi_l = \eta_l \phi$, where $\eta_l \equiv 0$ in $B(b_k, \frac{1}{2l})$, $\eta_l \equiv 1$ in $B(b_k, r) \setminus B(b_k, \frac{1}{l})$, $|\nabla \eta_l| \leq 4l$ when l is large. Recall the proof of (3.5), let $\phi_l = \eta_l \phi$ replace ϕ as to be the cut off function on $B(b_k, r)$, since ϕ_l vanish at the singular point b_k , as in the smooth case, we have (3.5):

$$\int (\phi_l)^2 |\nabla^2 G|^2 \leq C \left(\int (\phi_l)^2 |\nabla G|^2 + \int |\nabla(\phi_l)|^2 |\nabla G|^2 \right)$$

On the other hand,

$$\begin{aligned}
\int (\phi_l)^2 |\nabla^2 G|^2 &\leq C \left(\int (\phi_l)^2 |\nabla G|^2 + \int |\nabla(\phi_l)|^2 |\nabla G|^2 \right) \\
&\leq C \left(\int \phi^2 |\nabla G|^2 + \int |\nabla \phi|^2 |\nabla G|^2 + \int |\nabla \eta_l|^2 |\nabla G|^2 \right) \\
&\leq C \left(\int \phi^2 |\nabla G|^2 + \int |\nabla \phi|^2 |\nabla G|^2 + \sup |\nabla G|^2 \frac{1}{l^2} \right)
\end{aligned}$$

Letting l tends to ∞ , so we have

$$\int \phi^2 |\nabla^2 G|^2 \leq C \left(\int \phi^2 |\nabla G|^2 + \int |\nabla \phi|^2 |\nabla G|^2 \right)$$

and

$$\int_{B(b_k, \frac{r}{2})} |\nabla^2 G|^2 \leq C \frac{1}{r^2} \int_{B(b_k, r)} |\nabla G|^2 \leq Cr^2.$$

□

With lemma 4.3, even though we do not have the uniform bounded curvature across the singularity, the curvature condition $\sup_{S(r)} |Rm| \leq Cr^{-(2-\beta)}$ with $1 < \beta < 2$ is enough to construct $C^{1, \beta-1}$ coordinate around the singularity, see [BKN89] and [Tian90]. More precisely, we can construct coordinates $\Psi : S^3 \times (0, 1] \rightarrow \tilde{B}(b_k, 1)$ such that,

$$\Psi^* g_{ij}(x) - \delta_{ij} = O(|x|^\beta), \quad \partial \Psi^* g_{ij}(x) = O(|x|^{\beta-1}).$$

By [KD81], one can construct harmonic coordinates around the singularity with regularity at least $C^{1, \alpha}$, $\alpha = \beta - 1$. Now apply the Schauder theory on the coupled system (1.3)

$$\begin{cases} \Delta g = -2Ric + Q(g, \partial g) = -2\langle dG, dG \rangle + Q(g, \partial g) \\ \Delta G = dG * dG \end{cases} \quad (4.3)$$

under the harmonic coordinate. We first have that $g \in C^{1, \alpha}$ and $dG \in L^\infty$. By the second equation, $G \in C^{1, \alpha}$; going back to the first equation, the right hand side is C^α , therefore $g \in C^{2, \alpha}$; Go to the second equation again, $G \in C^{2, \alpha}$; and consequently, by the first equation again, $g \in C^{3, \alpha}$. Bootstrapping in this manner, we actually show that both g and G is smooth across the singularity. So we finish the proof of theorem 1.4. □

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